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# THE RING OF EVENLY WEIGHTED POINTS ON THE LINE

MILENA HERING AND BENJAMIN J. HOWARD

ABSTRACT. Let  $M_w = (\mathbb{P}^1)^n // \mathrm{SL}_2$  denote the geometric invariant theory quotient of  $(\mathbb{P}^1)^n$  by the diagonal action of  $\mathrm{SL}_2$  using the line bundle  $\mathcal{O}(w_1, w_2, \dots, w_n)$  on  $(\mathbb{P}^1)^n$ . Let  $R_w$  be the coordinate ring of  $M_w$ . We give a closed formula for the Hilbert function of  $R_w$ , which allows us to compute the degree of  $M_w$ . The graded parts of  $R_w$  are certain Kostka numbers, so this Hilbert function computes stretched Kostka numbers. If all the weights  $w_i$  are even, we find a presentation of  $R_w$  so that the ideal  $I_w$  of this presentation has a quadratic Gröbner basis. In particular,  $R_w$  is Koszul. We obtain this result by studying the homogeneous coordinate ring of a projective toric variety arising as a degeneration of  $M_w$ .

## 1. INTRODUCTION

The study of the ring of invariants for the action of the automorphism group of  $\mathbb{P}^1$  on  $n$  points on  $\mathbb{P}^1$  goes back to the 19th century. In 1894 Kempe [20] proved that this ring is generated by the invariants of lowest degree. More than a century later Howard, Millson, Snowden, and Vakil [17] were finally able to describe the ideal of relations between Kempe's generators, when the characteristic of the ground field  $\mathbf{k}$  is zero or  $p > 11$ .

More generally, for  $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$ , let  $L_w = \mathcal{O}_{(\mathbb{P}^1)^n}(w_1, \dots, w_n)$ . Assume that all  $w_i$  are positive, so that  $L_w$  is very ample. The group  $\mathrm{SL}(2)$  acts diagonally on  $(\mathbb{P}^1)^n$  and the line bundle  $L_w$  admits a unique linearization. Let

$$R_w = \left( \bigoplus_{d \geq 0} H^0((\mathbb{P}^1)^n, L_w^d) \right)^{\mathrm{SL}(2)}$$

denote the corresponding ring of invariant sections, and let  $M_w = (\mathbb{P}^1)^n // \mathrm{SL}(2)$  denote the GIT quotient. When  $w_i = 1$  for  $1 \leq i \leq n$ , we write  $w = 1^n$ .

In [16, Theorem 2.3] the authors show that  $R_w$  is generated by the invariants of lowest degree for arbitrary  $w$  and in [17, Theorem 1.1] that, in characteristic zero or  $p > 11$ , the ideal of relations  $I_w$  is generated by quadratic polynomials in the generators unless  $w = 1^6$ , in which case there is an essential cubic relation. Moreover, in [16, Section 2.15], the authors obtain a recursive formula for the degree of  $M_w$ .

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Our first theorem is an extension of Howe's formula [18, 5.4.2.3] for the Hilbert function of  $R_w$  in the case  $w = 1^n$  to arbitrary  $w$ . In particular, we obtain a closed formula for the degree of  $M_w$ .

**Theorem 1.1.** *Let  $[n] = \{1, \dots, n\}$ , and for  $J \subseteq [n]$ , set  $|w_J| = \sum_{j \in J} w_j$ ,  $w_\emptyset = 0$  and  $|w| = w_1 + \dots + w_n$ .*

(1) *The Hilbert function for  $R_w$  is given by*

$$h(d) = \sum_{\substack{J \subseteq [n] \\ |w_J| < |w|/2}} (-1)^{|J|} \binom{d(|w|/2 - |w_J|) + n - |J| - 2}{n - 2}$$

*if  $d|w|$  is even, and zero otherwise.*

(2) *For  $|w|$  even, the degree of  $M_w$  is*

$$\frac{1}{n - 2} \left( \sum_{\substack{J \subseteq [n] \\ |w_J| < |w|/2}} (-1)^{|J|} (|w|/2 - |w_J|)^{n-3} \left( \sum_{i=0}^{n-3} n - |J| - 2 - i \right) \right).$$

Let  $K(\lambda, \mu)$  be the Kostka number counting semistandard Young tableaux of shape  $\lambda$  with filling  $\mu$ . The dimension of the  $d$ -th graded part  $(R_w)_d$  of  $R_w$  is equal to the *stretched* Kostka number  $K(d\lambda, d\mu)$  where  $\lambda = (|w|/2, |w|/2)$  and  $\mu = w$  (see 2.1). We give a closed formula for Kostka numbers of this form in Proposition 3.2. It was shown in [22] and [2] that for partitions  $\lambda$  and  $\mu$  the function  $K(d\lambda, d\mu)$  is a polynomial in  $d$ . Thus the Hilbert function gives a closed formula for the polynomials  $K(d\lambda, d\mu)$  in this special case.

In [37] Jakub Witaszek studies the multigraded Poincaré-Hilbert series of the homogeneous coordinate ring of the Plücker embedding of the Grassmannian  $G(2, n)$  for a certain  $\mathbb{N}^n$ -grading. We obtain a closed formula for the multigraded Hilbert function and for the Poincaré-Hilbert series, see Remark 3.3.

For a field  $\mathbf{k}$ , recall that a graded  $\mathbf{k}$ -algebra  $R$  is *Koszul* if  $\mathbf{k}$  admits a linear free resolution as an  $R$ -module. If  $R = \mathbf{k}[X_0, \dots, X_N]/I$ , then the existence of a quadratic Gröbner basis for  $I$  implies that  $R$  is Koszul, which in turn implies that  $I$  is generated by quadratic equations. In [19], Keel and Tevelev show that the section ring of the log-canonical line bundle on  $\overline{M}_{0,n}$  is Koszul. However, while for  $w = 1^8$ ,  $I_w$  is generated by quadratic equations, we show in Example 3.9 that  $R_w$  is not Koszul.

In general, high enough Veronese subrings of graded rings are Koszul [1, 14], and up to a linear transformation, they admit a quadratic Gröbner basis [8]. We show that for  $R_w$  already the second Veronese subring satisfies these properties.

**Theorem 1.2.** *Assume  $w \in (2\mathbb{Z})^n$ . Then  $I_w$  admits a squarefree quadratic Gröbner basis. In particular,  $R_w$  is Koszul.*

Both theorems apply in all characteristics, and in fact more generally over the integers. Note that Theorem 1.2 implies that for  $w$  with  $|w|$  odd,  $I_w$  admits a squarefree quadratic Gröbner basis, since in this case  $R_w = R_{2w}$  by Proposition 2.1. In particular, if  $w = 1^n$  with  $n$  odd, then  $I_w$  admits a quadratic Gröbner basis. However, it is not known whether for  $w = 1^{10}$ ,  $R_w$  is Koszul, see Remark 3.10.

As in [16, 17], our proof is based on a toric degeneration. This toric degeneration is a SAGBI degeneration, and we show that this toric degeneration admits a quadratic Gröbner basis. After we shared our result with Manon, he was able to extend it to more general polytopes that arise as degenerations of the coordinate rings of the moduli stack of quasi-parabolic  $SL(2, \mathbb{C})$  principal bundles on a generic marked projective curve in [24, Theorem 1.10], see Remark 5.12.

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## 2. THE COORDINATE RING $R_w$ OF $M_w$

In this section we set up basic notation and describe the invariant ring  $R_w$  in terms of certain semistandard Young tableaux. Let  $\mathbf{k}$  be a field, and let

$$S = \mathbf{k}[x_1, y_1, x_2, y_2, \dots, x_n, y_n],$$

which we view as the set of polynomial functions on the space  $\mathbb{A}^{2 \times n}$  of  $2 \times n$  matrices with entries in the field  $\mathbf{k}$ :

$$\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix}.$$

The polynomial ring  $S$  is graded by  $\mathbb{N}^n$ , where the degree of the monomial  $\prod_{i=1}^n x_i^{a_i} y_i^{b_i}$  is equal to  $(a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$ . Given  $r = (r_1, \dots, r_n) \in \mathbb{N}^n$ , let  $S_r$  denote the  $r$ -th graded part of  $S$ . Let  $L_r = \mathcal{O}(r_1, \dots, r_n)$ . Viewing  $x_i, y_i$  as homogeneous coordinates for the  $i$ 'th point in  $(\mathbb{P}^1)^n$ , we have  $H^0((\mathbb{P}^1)^n, L_r) = S_r$ . The line bundle  $L_r$  admits a linearization for the diagonal action of  $SL(2, \mathbf{k})$  on  $(\mathbb{P}^1)^n$ , (see [28, Chapter 3.1]) such that the induced action on the section ring  $R(L_r)$  is given by matrix multiplication on the left. It is easy to see that for  $i < j$  the polynomials

$$p_{ij} = \det \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}$$

are invariant under the  $SL(2, \mathbf{k})$  action. Note that  $p_{i,j}$  are the Plücker coordinates on the Grassmannian  $G(2, n)$ . The First Fundamental Theorem of Invariant Theory says that they generate the ring of invariants  $S^{SL(2, \mathbf{k})}$ , [6, Theorem 2.1]. Note that  $S^{SL(2, \mathbf{k})}$  is the homogeneous coordinate ring of  $G(2, n)$  in the Plücker embedding.

For our purposes, it is most convenient to study this invariant ring using tableaux of shape  $(k, k)$ ,

$$(2.1) \quad \tau = \begin{array}{|c|c|} \hline i_1 & i_2 \\ \hline j_1 & j_2 \\ \hline \end{array} \cdots \begin{array}{|c|} \hline i_k \\ \hline j_k \\ \hline \end{array},$$

where  $1 \leq i_\ell, j_\ell \leq n$ . A tableau  $\tau$  is called *semistandard* if its entries are weakly increasing in the rows and strictly increasing in the columns, i.e., in our setting we have  $i_1 \leq \cdots \leq i_k, j_1 \leq \cdots \leq j_k$ , and  $i_1 < j_1, \dots, i_k < j_k$ . The *content* of a tableau  $\tau$  is the  $n$ -tuple  $w(\tau) = (w(\tau)_1, \dots, w(\tau)_n)$ , where  $w(\tau)_i$  denotes the number of times  $i$  occurs in  $\tau$ .

To a semistandard tableau  $\tau$  as in (2.1) we associate the polynomial

$$s_\tau = p_{i_1, j_1} p_{i_2, j_2} \cdots p_{i_k, j_k} \in S^{\text{SL}(2, \mathbf{k})}.$$

Note that  $s_\tau$  is homogeneous of degree  $w(\tau)$ .

The following proposition is an algebraic incarnation of the Gel'fand-MacPherson correspondence [11]. Let  $R_w$  be the invariant ring of the introduction and let  $|w| = w_1 + \cdots + w_n$ .

**Proposition 2.1.** *The polynomials  $s_\tau$ , where  $\tau$  ranges over all semistandard tableaux of shape  $(\frac{d|w|}{2}, \frac{d|w|}{2})$  with content  $dw$  for some  $d \in \mathbb{N}$ , form a vector space basis for the invariant ring  $R_w$ .*

*Proof.* Let  $X = (\mathbb{P}^1)^n$ . The torus  $T_w = \{\text{diag}(t_1, \dots, t_n) \in \text{GL}(n, \mathbf{k}) \mid t_1^{w_1} t_2^{w_2} \cdots t_n^{w_n} = 1\}$  acts on the right of  $\mathbb{A}^{2 \times n}$  by matrix multiplication inducing an action on  $S$  such that  $S^{T_w} = \oplus S_{dw}$ . Thus we obtain

$$R_w = \left( \bigoplus_d H^0((\mathbb{P}^1)^n, L_w^d) \right)^{\text{SL}(2, \mathbf{k})} = \left( \bigoplus_d S_{dw} \right)^{\text{SL}(2, \mathbf{k})} = (S^{T_w})^{\text{SL}(2, \mathbf{k})}.$$

Since the actions of  $\text{SL}(2, \mathbf{k})$  and  $T_w$  commute, we have  $(S^{T_w})^{\text{SL}(2, \mathbf{k})} = (S^{\text{SL}(2, \mathbf{k})})^{T_w}$ . Now,  $S^{\text{SL}(2, \mathbf{k})}$  has a vector space basis consisting of  $s_\tau$  where  $\tau$  ranges over semistandard Young tableaux of shape  $(k, k)$ , see for example [6, Theorem 2.3]. Let  $f = \sum_\tau a_\tau s_\tau \in S^{\text{SL}(2, \mathbf{k})}$ , where  $\tau$  runs over semistandard tableaux. Since the action of  $T_w$  is linear, and the  $s_\tau$  are linearly independent,  $f$  is invariant under  $T_w$  if and only if every  $s_\tau$  is invariant. Note that for a tableau  $\tau$  with content  $w(\tau)$ , we have  $t \cdot s_\tau = t_1^{w(\tau)_1} \cdots t_n^{w(\tau)_n} s_\tau$ . In particular,  $s_\tau$  is invariant if and only if there is  $d$  with  $w(\tau) = dw$ . The claim follows.  $\square$

In particular, when  $w = 1^n$  and  $n = 2m$  even, the dimension of the space of lowest degree invariants in  $R_w$  is the Catalan number  $C_m$ .

**Remark 2.2.** One can view  $R_w$  as a multigraded Veronese subring of the Plücker algebra. The Plücker algebra  $S^{\text{SL}(2, \mathbf{k})}$  admits a  $\mathbb{N}^n$ -grading determined by the weight under the action of the diagonal torus  $T = \text{diag}(t_1, \dots, t_n)$  via  $t \cdot p_{i,j} = t_i t_j p_{i,j}$ . Then for a tableau  $\tau$  with content  $w(\tau)$ , we have  $t \cdot s_\tau = t^{w(\tau)} s_\tau$  and thus we can conclude as in the proof of Proposition 2.1 that

$$(S^{\text{SL}(2, \mathbf{k})})_w = \langle \{s_\tau \mid \tau \text{ is semistandard of shape } (|w|/2, |w|/2) \text{ with filling } w(\tau) = w\} \rangle.$$

Here  $\langle \cdots \rangle$  denotes the span as a vector space over  $\mathbf{k}$ . In particular,  $(S^{\text{SL}(2, \mathbf{k})})_w = 0$  if  $|w|$  is odd. It then follows that  $R_w = \bigoplus_{d \in \mathbb{N}} (S^{\text{SL}(2, \mathbf{k})})_{dw}$ .

**Definition 2.3.** To a tableau  $\tau$  of shape  $(k, k)$  is associated a partition  $\nu_\tau = (\nu_1, \dots, \nu_n)$  of  $d$ , the content of the first row of  $\tau$ , i.e.,  $\nu_i = |\{j \mid i_j = i\}|$ .

The following Lemma can be easily deduced from the discussion in [15, Section 3]. We include a sketch of the proof for the convenience of the reader.

**Lemma 2.4.** *The semistandard Young tableaux of shape  $(|w|/2, |w|/2)$  with filling  $w$  are in bijection with partitions  $\nu = (\nu_1, \dots, \nu_n)$  of  $|w|/2$  satisfying*

$$(2.2) \quad 0 \leq \nu_\ell \leq w_\ell \text{ and}$$

$$(2.3) \quad 2(\nu_1 + \cdots + \nu_{\ell-1}) + \nu_\ell \geq w_1 + \cdots + w_\ell$$

for  $1 \leq \ell \leq n$ . These conditions imply  $\nu_1 = w_1$  and  $\nu_n = 0$ .

*Proof.* To a tableau with filling  $w$  and increasing rows is associated a partition of  $|w|/2$  by Definition 2.3. Conversely, to a partition  $\nu$  satisfying (2.2), we associate a tableau  $\tau$  of shape  $(\frac{|w|}{2}, \frac{|w|}{2})$  by filling the first row with  $\nu_1$  1's,  $\nu_2$  2's, etc., and the second row with  $(w_1 - \nu_1)$  1's,  $(w_2 - \nu_2)$  2's, etc. By construction the rows of this tableau are increasing and it has filling  $w$ . These associations are inverse to each other. Moreover,  $\tau$  is semistandard if and only if  $\nu_1 + \cdots + \nu_{\ell-1} \geq (w_1 - \nu_1) + \cdots + (w_\ell - \nu_\ell)$  for  $1 \leq \ell \leq n$ . This condition is equivalent to (2.3).  $\square$

### 3. THE HILBERT POLYNOMIAL AND DEGREE OF $M_w$

In this section we will prove the formulas for the Hilbert function of  $R_w$  and the degree of  $M_w$  of Theorem 1.1. Our techniques are similar to those of Howe [18, 5.4.2.3.] who computed the case when  $w = 1^n$ .

Let  $\lambda = (|w|/2, |w|/2)$  and  $\mu = w$ . For partitions  $\lambda$  and  $\mu$  the Kostka numbers  $K(\lambda, \mu)$  are defined to be the number of semistandard Young tableaux of shape  $\lambda$  and content  $\mu$ . For a partition  $\lambda = (\lambda_1, \dots, \lambda_s)$ , we let  $d\lambda = (d\lambda_1, \dots, d\lambda_s)$ . Note that by Proposition 2.1 the dimension of  $(R_w)_d$  is equal to  $K(d\lambda, d\mu)$ .

In order to compute the Hilbert polynomial, we give a formula for these particular Kostka numbers. The main step in proving this formula is to set up a relationship between the Kostka numbers and numbers of the corresponding partitions of Definition 2.3. We let  $\Pi(n, \infty, k) = \{(\nu_1, \dots, \nu_n) \mid 0 \leq \nu_i \text{ for } i \in [n] \text{ and } \nu_1 + \cdots + \nu_n = k\}$ , and let

$$(3.1) \quad \pi(n, \infty, k) = |\Pi(n, \infty, k)| = \binom{n-1+k}{n-1}.$$

Let  $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ . We let  $\Pi(n, w, k) = \{(\nu_1, \dots, \nu_n) \mid 0 \leq \nu_i \leq w_i \text{ for } i \in [n] \text{ and } \nu_1 + \cdots + \nu_n = k\}$  and let  $\pi(n, w, k) = |\Pi(n, w, k)|$ . For a subset  $I \subseteq [n] = \{1, \dots, n\}$ , we let  $\Pi(n, w_I, k) = \{(\nu_1, \dots, \nu_n) \in \Pi(n, \infty, k) \mid 0 \leq \nu_i \leq w_i \text{ for } i \in I\}$  and  $\pi(n, w_I, k) = |\Pi(n, w_I, k)|$ .

**Lemma 3.1.** *For any  $I \subseteq [n]$ ,*

$$\begin{aligned} \pi(n, w_I, k) &= \sum_{J \subseteq I} (-1)^{|J|} \pi(n, \infty, k - (|w_J| + |J|)) \\ &= \sum_{J \subseteq I} (-1)^{|J|} \binom{n-1+k-(|w_J|+|J|)}{n-1}. \end{aligned}$$

*Proof.* We proceed by induction on the cardinality of  $I$ . When  $I = \emptyset$ , the above claim is immediate. If  $I \neq \emptyset$ , let  $j \in I$ . Since

$$\pi(n, w_I, k) = \pi(n, w_{I \setminus \{j\}}, k) - \pi(n, w_{I \setminus \{j\}}, k - w_j - 1),$$

the claim follows from the induction hypothesis. The last equality follows from (3.1).  $\square$

**Proposition 3.2.** *Let  $w \in \mathbb{N}^n$  and assume that  $|w|$  is even. Then for  $\lambda = (|w|/2, |w|/2)$  and  $\mu = w$ , we have*

$$\begin{aligned} K(\lambda, \mu) &= \pi(n, w, |w|/2) - \pi(n, w, |w|/2 - 1) \\ &= \sum_{\substack{J \subseteq [n] \\ |w_J| < |w|/2}} (-1)^{|J|} \binom{|w|/2 - |w_J| + n - |J| - 2}{n-2} \end{aligned}$$

*Proof.* For the first equality, we need to express the Kostka numbers in terms of partitions. For  $\nu \in \mathbb{R}^n$ , we define a function  $f_\nu: [n] \rightarrow \mathbb{R}$  by

$$(3.2) \quad f_\nu(i) = 2\nu_1 + \cdots + 2\nu_{i-1} + \nu_i - (w_1 + \cdots + w_i)$$

for  $1 \leq i \leq n$ . Then Lemma 2.4 implies that for  $w \in \mathbb{Z}^n$  with  $|w|$  even,  $\lambda = (|w|/2, |w|/2)$ , and  $\mu = w$  we have

$$(3.3) \quad K(\lambda, \mu) = |\{\nu \in \Pi(n, w, |w|/2) \mid f_\nu(i) \geq 0 \text{ for all } 1 \leq i \leq n\}|.$$

For  $\nu \in \mathbb{Z}^n$  we let  $m_\nu = \min\{f_\nu(j) \mid j \in [n]\}$  and  $i_\nu = \max\{j \mid f_\nu(j) = m_\nu\}$  and define

$$\begin{aligned} \phi: \{\nu \in \Pi(n, w, |w|/2) \mid \exists i \text{ such that } f_\nu(i) < 0\} &\rightarrow \Pi(n, w, |w|/2 - 1) \\ (\nu_1, \dots, \nu_n) &\mapsto (\nu_1, \dots, \nu_{i_\nu} - 1, \dots, \nu_n). \end{aligned}$$

We claim that  $\phi$  is well-defined and gives a bijection. Note that the first equality then follows from this claim together with (3.3).

The following equalities follow easily from the definition of  $f_\nu$ :

$$(3.4) \quad f_\nu(i+1) = f_\nu(i) + \nu_i - w_{i+1} + \nu_{i+1}$$

$$(3.5) \quad f_\nu(n) = 2|\nu| - |w| - \nu_n.$$

To see that  $\phi$  is well defined, we have to show that  $\nu_{i_\nu} > 0$ . When  $i_\nu < n$  then  $f_\nu(i+1) > f_\nu(i)$  and (3.4) implies  $\nu_i > w_{i+1} - \nu_{i+1} \geq 0$ . If  $i_\nu = n$ , then  $f_\nu(n) = m_\nu < 0$ , since there exists  $i$  such that  $f_\nu(i) < 0$  and  $m_\nu \leq f_\nu(i)$ . Since  $|\nu| = |w|/2$ ,  $f_\nu(n) = -\nu_n$  by (3.5), so we have  $\nu_n > 0$ .

To see that  $\phi$  is a bijection, we exhibit the inverse map. For  $\nu' \in \Pi(n, w, |w|/2 - 1)$ , we let  $j_{\nu'} = \min\{i \mid f_{\nu'}(i) = m_{\nu'}\}$  and define

$$\psi: (\nu'_1, \dots, \nu'_n) \mapsto (\nu'_1, \dots, \nu'_{j_{\nu'}} + 1, \dots, \nu'_n).$$

We have

$$(3.6) \quad f_{\psi(\nu')}(i) = \begin{cases} f_{\nu'}(i) & \geq m_{\nu'} + 1 & \text{if } i < j_{\nu'} \\ f_{\nu'}(i) + 1 & = m_{\nu'} + 1 & \text{if } i = j_{\nu'} \\ f_{\nu'}(i) + 2 & \geq m_{\nu'} + 2 & \text{if } i > j_{\nu'}. \end{cases}$$

We have to show that  $\psi$  is well-defined. If  $j_{\nu'} > 1$ , then  $f_{\psi(\nu')}(j_{\nu'}) = f_{\psi(\nu')}(j_{\nu'} - 1) + \nu'_{j_{\nu'}-1} - w_{j_{\nu'}} + \nu'_{j_{\nu'}} + 1$  by (3.4). Plugging in the values from (3.6), we see that  $w_{j_{\nu'}} \geq \nu'_{j_{\nu'}} + 1$ . If  $j_{\nu'} = 1$ , then  $f_{\psi(\nu')}(1) = \nu'_1 + 1 - w_1 = m_{\nu'} + 1$ . However, note that  $m_{\nu'} \leq f_{\nu'}(n) \leq -2$  by 3.5, and so it follows that  $\nu'_1 + 2 \leq w_1$ . Moreover, (3.6) implies that  $i_{\psi(\nu')} = j_{\nu'}$ . One can check similarly that  $j_{\phi(\nu)} = i_{\nu}$  and it follows that  $\phi$  and  $\psi$  are inverse to each other.

For the second equality, note that Lemma 3.1 implies that

$$\begin{aligned} & \pi(n, w, |w|/2) - \pi(n, w, |w|/2 - 1) \\ &= \sum_{J \subseteq [n]} (-1)^{|J|} \left[ \binom{n-1+|w|/2-(|w_J|+|J|)}{n-1} - \binom{n-2+|w|/2-(|w_J|+|J|)}{n-1} \right] \end{aligned}$$

Using the identity  $\binom{m}{n} - \binom{m-1}{n} = \binom{m-1}{n-1}$ , one obtains the formula in the statement. Note that if  $|w_J| \geq |w|/2$ , the expression in the top of the binomial coefficient is less than  $n-2$ , so it suffices to sum over those  $J \subseteq [n]$  such that  $|w_J| < |w|/2$ .  $\square$

*Proof of Theorem 1.1.* Fix  $w \in \mathbb{N}^n$ , let  $\lambda = (|w|/2, |w|/2)$ , and let  $\mu = w$ . It follows from Proposition 2.1 that  $\dim(R_w)_d = 0$  if  $d|w|$  is odd and that  $\dim(R_w)_d = K(d\lambda, d\mu)$  if  $d|w|$  is even. The formula for  $h(d)$  then follows from Proposition 3.2.

The formula for the degree of  $M_w$  is obtained by computing the coefficient of  $d^{n-3}$  in the Hilbert polynomial and multiplying by  $(n-3)!$ .  $\square$

**Remark 3.3.** Our formula also implies a closed formula for the multigraded Hilbert function and Poincaré-Hilbert series of the coordinate ring of the Grassmannian  $G(2, n)$  in the Plücker embedding with the multigrading described in Remark 2.2. In [37, Theorem 3.4.3] Jakub Witaszek gives a recursive formula for the multigraded Poincaré-Hilbert series  $\sum_{w \in \Lambda} \dim(S^{\text{SL}(2, \mathbf{k})})_w z^w$  of the Plücker algebra for the multigrading described in Remark 2.2. He also obtains a combinatorial formula for the Poincaré-Hilbert series.

Let  $\Lambda = \{w \in \mathbb{Z}^n \mid |w| \in 2\mathbb{Z}\}$ . By Remark 2.2, we have  $\dim(S^{\text{SL}(2, \mathbf{k})})_w = K(\lambda, \mu)$ , where  $\lambda = (|w|/2, |w|/2)$  and  $\mu = w$ . It follows that the support of the multigraded Hilbert function  $\mathbf{h}$  is  $\{w \in \Lambda \mid |w|/2 \geq w_j \text{ for all } 1 \leq j \leq n\}$ . Then Proposition 3.2 implies that for  $w \in \Lambda$ ,

$$\mathbf{h}(w) = \dim(S^{\text{SL}(2, \mathbf{k})})_w = \sum_{\substack{J \subseteq [n] \\ |w_J| < |w|/2}} (-1)^{|J|} \binom{|w|/2 - |w_J| + n - |J| - 2}{n-2}.$$

So we obtain a closed formula for the multigraded Poincaré-Hilbert series.

A point  $p = (p_1, \dots, p_n) \in (\mathbb{P}^1)^n$  is stable (resp. semistable) for the  $\text{SL}(2, \mathbf{k})$ -linearization of  $L_w$  if for all subsets of indices of colliding points  $J = \{j \in [n] \mid p_j = p \text{ for some } p\}$  we have  $|w_J| < |w|/2$  (resp.  $|w_J| \leq |w|/2$ ), see [28, Chapter 3], [36, Section



6], or [13, Section 8]. Identifying  $\mathbb{N}^n$  with the effective divisors on  $(\mathbb{P}^1)^n$ , we see that the fact that in the formula for the multigraded Hilbert function we sum over those  $J \subseteq [n]$  such that  $|w_J| < |w|/2$  reflects the chamber structure for the GIT chambers whose walls are given by  $|w_J| = |w|/2$ . In particular, the multigraded Hilbert function is piecewise polynomial in  $w \in \Lambda$  and the domains of polynomiality agree with the GIT chambers.

**Remark 3.4.** In the formula for the Hilbert polynomial of Theorem 1.1, the terms of degree  $(n-2)$  cancel out since  $\dim M_w = n-3$ . Thus we obtain the following identity:

$$\sum_{J \subseteq [n]} (-1)^{|J|} (|w|/2 - |w_J|)^{n-2} = 0.$$

**Remark 3.5.** Let  $g(d)$  denote the Hilbert function of  $G(2, n)$  in the Plücker embedding. Recall the multigraded Hilbert function  $\mathbf{h}$  for the Plücker embedding from Remark 3.3. Then we have

$$g(d) = \sum_{\substack{w \in \mathbb{N}^n \\ |w| = 2d}} \mathbf{h}(w),$$

implying the identity

$$(3.7) \quad \binom{n+d-1}{d}^2 - \binom{n+d}{d+1} \binom{n+d-2}{d-1} \\ = \sum_{\substack{w \in \mathbb{N}^n \\ |w| = 2d}} \sum_{\substack{J \subseteq [n] \\ |w_J| < |w|/2}} (-1)^{|J|} \binom{|w|/2 - |w_J| + n - |J| - 2}{n-2}.$$

**Remark 3.6.** While our formula counts semistandard tableaux, it contains negative signs. It would be nice to have a formula with positive coefficients. In fact, King, Tollu and Toumazet conjecture that for arbitrary  $\lambda, \mu$  the coefficients of the polynomial  $K(d\lambda, d\mu)$  are positive in [21, Conjecture 3.2].

**Remark 3.7.** Narayanan shows in [29, Theorem 1] that the problem of computing Kostka numbers  $K(\lambda, \mu)$  is  $\#P$ -complete. Note that for our formula, one has to compute first all subsets of  $[n]$ , where  $n$  is the length of  $\mu$ .

**Example 3.8.** The formula in Theorem 1.1 shows that  $\deg(M_4) = 1$ ,  $\deg(M_6) = 3$ ,  $\deg(M_8) = 40$ ,  $\deg(M_{10}) = 1225$ ,  $\deg(M_{12}) = 67956$ ,  $\deg(M_{14}) = 5986134$ ,  $\deg(M_{16}) = 769550496$ , so this sequence agrees with A012250 on Sloane's online encyclopedia of integer sequences [32], compare [16, Section 2.15]. Similarly, when  $w = (2, \dots, 2)$ , the degrees of  $M_w$  are  $\deg(M_{24}) = 2$ ,  $\deg(M_{25}) = 5$ ,  $\deg(M_{26}) = 24$ ,  $\deg(M_{27}) = 154$ ,  $\deg(M_{28}) = 1280$ ,  $\deg(M_{29}) = 13005$ ,  $\deg(M_{210}) = 156800$ ,  $\deg(M_{211}) = 2189726$  which agrees with sequence A012249 [33].

The following example shows that while for  $w = 1^8$ ,  $I_w$  is generated by quadratic equations by [17], the ring of invariants  $R_w$  is not Koszul.

**Example 3.9.** Recall that for a graded algebra  $R$  the Hilbert series is given by  $H(z) = \sum_{d=0}^{\infty} \dim(R_d) z^d$ . Similarly, the Poincaré series is given by  $P(z) = \sum_{i=0}^{\infty} \dim \operatorname{Tor}_i^R(\mathbf{k}, \mathbf{k}) z^i$ . Let  $\mathbf{P}(u, v) = \sum_{i=0}^{\infty} \dim \operatorname{Tor}_i^R(\mathbf{k}, \mathbf{k})_j u^i v^j$ . Then we have  $H(z) \mathbf{P}(-1, z) = 1$  by [31, Chapter

2, Proposition 2.1]. If  $R$  is Koszul, then the minimal free graded resolution of  $\mathbf{k}$  over  $R$  is linear, and so  $P(uv) = \mathbf{P}(u, v)$ . Then  $H(z)P(-z) = 1$  if and only if  $R$  is Koszul, see [10, Theorem 1]. In particular, the power series  $H(-z)^{-1} = P(z)$  must have positive coefficients.

Note that for  $w = 1^8$ , the Hilbert function is given by

$$h(d) = \sum_{j=0}^3 (-1)^j \binom{8}{j} \binom{d(4-j)+6-j}{6},$$

so the Hilbert series is

$$\begin{aligned} H(z) &= 1 + 14z + 91z^2 + 364z^3 + 1085z^4 + 2666z^5 + 5719z^6 \\ &\quad + 11096z^7 + 19929z^8 + O(z^9) \\ &= \frac{1 + 8z + 22z^2 + 8z^3 + z^4}{(1-z)^6}. \end{aligned}$$

Then

$$(3.8) \quad H(-z)^{-1} = 1 + 14z + 105z^2 + 560z^3 + 2296z^4 + 6880z^5 + 8904z^6 \\ - 62320z^7 - 641704z^8 + O(z^9).$$

**Remark 3.10.** For  $w = 1^{10}$  it is not known whether  $R_w$  is Koszul or whether the ideal of relations between the generators admits a quadratic Gröbner basis. One can check that the first 800 coefficients of  $H(-z)^{-1}$  are positive, however it is not known whether the relation  $H(-z)P(z) = 1$  is satisfied. In general one might hope that for  $n$  large and  $w = 1^n$  the ring  $R_w$  is Koszul or has other nice properties, such as Green's property  $N_p$ , see for example [23, Section 1.8.D].

#### 4. A SAGBI DEGENERATION OF $R_w$

A crucial part in the proof of [17, Theorem 1.1] is the existence of toric degenerations of  $M_w$  indexed by trivalent trees on  $n$  leaves, see [17, Section 3.3]. The existence of one of these toric degenerations had been established by Foth and Hu in [9, Theorem 3.2]. In fact, the toric degenerations are torus quotients of the toric degenerations of the Grassmannian  $G(2, n)$  studied by Sturmfels and Speyer [34] and by Gonciulea and Lakshmibai in [12]. Manon mentions in [26, Theorem 1.3.6] that the ring of invariants  $R$  admits a SAGBI degeneration. In this section we will give an explicit description of this SAGBI degeneration, by taking the torus invariants of the SAGBI degeneration described in [27, Section 14.3].

Let  $R$  be a finitely generated subalgebra of the polynomial ring  $S = \mathbf{k}[x_1, \dots, x_n]$ , and let  $\prec$  be a term order on  $S$ . Let  $\text{in}_{\prec}(R)$  be the subalgebra of  $S$  generated by the initial terms of the elements of  $R$ . Assume this subalgebra is finitely generated. (Note that this is rarely the case). A set of generators  $\{f_1, \dots, f_r\}$  is called a *SAGBI basis* for  $R$  with respect to  $\prec$  if  $\text{in}_{\prec}f_1, \dots, \text{in}_{\prec}f_r$  generate  $\text{in}_{\prec}(R)$ . Let  $I$  be the ideal of relations between the generators of  $R$ . Then  $\text{in}_{\prec}(R) \cong R/\text{in}_{\prec}I$ , see [4, Proof of Corollary 2.1]. Since  $\text{in}_{\prec}(R)$  is a monomial algebra, it is toric, and therefore the existence of a SAGBI basis for an

algebra implies the existence of a flat degeneration of this algebra to a toric algebra [7, Theorem 15.17]. Note that the toric algebra need not be normal.

Now let  $S = \mathbf{k}[x_1, \dots, x_n, y_1, \dots, y_n]$  and  $R$  the ring of invariants as in Section 2. Let  $\prec$  be the purely lexicographic term order with  $x_1 \succ \dots \succ x_n \succ y_1 \succ \dots \succ y_n$ .

Recall the polynomial  $s_\tau$  associated to a  $2 \times n$  tableau  $\tau = \begin{array}{|c|c|} \hline i_1 & i_2 \\ \hline j_1 & j_2 \\ \hline \end{array} \cdots \begin{array}{|c|} \hline i_r \\ \hline j_r \\ \hline \end{array}$ . We also associate a monomial  $m_\tau = x_{i_1} \cdots x_{i_r} y_{j_1} \cdots y_{j_r} \in S$ .

**Lemma 4.1.** *A monomial  $m$  is a leading monomial of an element in  $R$  if and only if  $m = m_\tau$ , where  $\tau$  is semistandard with filling  $dw$ , for some  $d$ . In particular, the set of these  $m_\tau$  is a vector space basis for  $\text{in}_\prec(R)$ .*

*Proof.* By Proposition 2.1, the polynomials  $s_\tau$ , where  $\tau$  ranges over semistandard Young tableaux of shape  $2 \times \frac{d|w|}{2}$  with content  $dw$  form a vector space basis of  $R$ . Note that every monomial occurring in  $s_\tau$  is of the form  $m_{\tau'}$ , where  $\tau'$  is a (not necessarily semistandard) tableau with the same shape and same content as  $\tau$ . Among those monomials, the largest with respect to the term order  $\prec$  is  $m_\tau$ . In particular,  $m_\tau \in \text{in}_\prec(R)$ . Moreover, the leading monomial of an element  $\sum_{\tau \in T} a_\tau s_\tau \in R$  is  $m_{\tau'}$ , where  $m_{\tau'}$  is the largest monomial in  $\{m_\tau \mid \tau \in T, a_\tau \neq 0\}$  with respect to the term order  $\prec$ . Since to distinct semistandard tableaux are associated distinct monomials, the  $m_\tau$  are also linearly independent. See also [27, Lemma 14.13].  $\square$

**Definition 4.2.** Let  $w \in \mathbb{N}^n$ . Let  $Q_w \subset \mathbb{R}^n$  be the polytope defined by  $\nu_1 + \dots + \nu_n = \frac{|w|}{2}$  and the inequalities (2.2) and (2.3).

**Example 4.3.** Let  $w = (2, 2, 2, 2, 2)$ . Then the inequalities for  $Q_w$  imply  $\nu_1 = 2, \nu_5 = 0$ , and  $\nu_4 = 3 - \nu_2 - \nu_3$ . Thus  $Q_w$  is the 2-dimensional polytope given by the inequalities  $0 \leq \nu_2, \nu_3 \leq 2, 0 \leq 3 - \nu_2 - \nu_3 \leq 2$ , and  $2\nu_2 + \nu_3 \geq 2$ . From this we can see that  $Q_w$  is isomorphic to the convex hull of  $\langle (1, 0), (2, 0), (2, 1), (1, 2), (0, 2) \rangle$ .

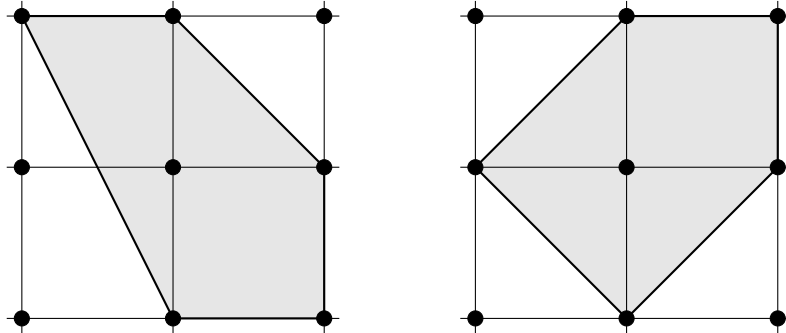


FIGURE 1. The polytopes  $Q_w$  of Example 4.3 (with reference lattice  $\mathbb{Z}^n$ ) and  $P_w$  of Example 5.2 (with reference lattice  $(2\mathbb{Z})^n$ ) for  $w = (2, 2, 2, 2, 2)$ .

Note that when  $w = (1, 1, 1, 1, 1)$ ,  $Q_w$  contains no lattice points; in particular, it is not a lattice polytope. However, it follows from Lemma 5.3 and Lemma 5.4 that  $2Q_w$  is a lattice polytope for all  $w$ .

**Remark 4.4.** Note that  $Q_w$  is a Gelfand-Tsetlin polytope, see for example [5].

Recall that to a rational polytope  $P$  is associated a graded monoid  $S_P = \{(\mathbf{u}, d) \mid \mathbf{u} \in dP \cap \mathbb{Z}^n, d \in \mathbb{N}\}$  and an algebra  $\mathbf{k}[S_P] = \langle x^{\mathbf{u}} z^d \mid (\mathbf{u}, d) \in S_P \rangle$ .

**Proposition 4.5.** *The subalgebra  $\text{in}_{\prec} R$  is isomorphic to the polytopal semigroup algebra  $\mathbf{k}[S_{Q_w}]$ . In particular,  $\text{in}_{\prec} R$  is finitely generated.*

*Proof.* By Lemma 4.1,  $\text{in}_{\prec}(R)$  is generated as a vector space by  $m_{\tau}$  where  $\tau$  runs over all semistandard Young tableaux of shape  $2 \times \frac{d|w|}{2}$  with content  $dw$ . Note that for such a semistandard Young tableau  $\tau$  we have  $m_{\tau} = x^{\nu} y^{w-\nu}$ , where  $\nu$  is the partition associated to  $\tau$  as in Definition 2.3. Let  $\phi: \text{in}_{\prec} R \rightarrow \mathbf{k}[x_1, \dots, x_n, z]$  be the homomorphism induced by letting  $\phi(m_{\tau}) = x^{\nu} z^d$ , when  $\tau$  has shape  $2 \times \frac{d|w|}{2}$ . Since  $\nu$  determines  $\tau$ , this homomorphism is injective. It follows from Lemma 2.4 that it is surjective onto  $\mathbf{k}[S_{Q_w}]$ . Note that for any rational polytope, the associated polytopal semigroup algebra  $\mathbf{k}[S_{Q_w}]$  is finitely generated.  $\square$

**Remark 4.6.** When  $w = 1^n$ , one can show that this toric degeneration is a degeneration of Fano varieties, and the corresponding line bundle the anticanonical line bundle. As this seems well known, we omit the proof.

## 5. THE QUADRATIC GRÖBNER BASIS

We now assume that  $w \in (2\mathbb{Z})^n$ . The goal of this section is to show that in this case, the polytopal semigroup algebra  $\mathbf{k}[S_{Q_w}]$  is generated in degree 1, and admits a presentation such that the ideal of relations has a quadratic Gröbner basis. It then follows from general properties of SAGBI degenerations that  $R_w$  also admits such a presentation.

Instead of showing these properties directly for the polytopes  $Q_w$ , we will show them for a family of isomorphic polytopes  $P_w$ . The latter ones exhibit more symmetry that we will exploit later on. We denote the  $i$ 'th component of a vector  $u \in \mathbb{R}^{n-3}$  by  $u(i+1)$  instead of  $u_i$ .

**Definition 5.1.** We say that a point  $(x, y, z) \in \mathbb{R}^3$  satisfies the triangle inequalities if  $x + y \geq z$ ,  $x + z \geq y$ , and  $y + z \geq x$ . To  $w \in (2\mathbb{N})^n$  is associated a polytope  $P_w \subset \mathbb{R}^{n-3}$  consisting of  $(u(2), \dots, u(n-2))$  such that

$$(w_1, w_2, u(2)), (u(n-2), w_{n-1}, w_n), \text{ and } (u(i-1), w_i, u(i))$$

satisfy the triangle inequalities for  $3 \leq i \leq n-2$ .

**Example 5.2.** When  $w = (2, 2, 2, 2, 2)$ , the polytope  $P_w$  is given by the inequalities  $0 \leq u(2) \leq 4$ ,  $0 \leq u(3) \leq 4$ ,  $u(2) + u(3) \geq 2$ ,  $u(2) + 2 \geq u(3)$ ,  $u(3) + 2 \geq u(2)$ . See Figure 4.3.

It follows from Lemma 5.4 that  $P_w$  is a lattice polytope for the lattice  $M := (2\mathbb{Z})^{n-3}$ .

**Lemma 5.3.** *The polytopal semigroups  $S_{Q_w} = \{(\nu, d) \mid d \in \mathbb{N}, \nu \in dQ_w \cap \mathbb{Z}^n\}$  and  $S_{P_w} = \{(u, d) \mid d \in \mathbb{N}, u \in dP_w \cap M\}$  are isomorphic.*

*Proof.* Let  $V = \{(\nu_1, \dots, \nu_n, d) \in \mathbb{R}^n \times \mathbb{R} \mid \nu_1 = dw_1, \nu_n = 0, \text{ and } \nu_1 + \dots + \nu_n = \frac{d|w|}{2}\}$ , an affine subspace of  $\mathbb{R}^n \times \mathbb{R}$ . Then  $S_{Q_w} \subset V$ . We identify  $V$  with  $\mathbb{R}^{n-3} \times \mathbb{R}$  via  $(\nu_1, \dots, \nu_n, d) \mapsto (\nu_2, \dots, \nu_{n-2}, d)$ , with inverse  $(\nu_2, \dots, \nu_{n-2}, d) \mapsto (dw_1, \nu_2, \dots, \nu_{n-2}, \frac{d|w|}{2} - dw_1 - \nu_2 - \dots - \nu_{n-2}, 0, d)$ . Since  $|w|$  is even, this identification respects the lattices  $V \cap (\mathbb{Z}^n \times \mathbb{Z})$  and  $\mathbb{Z}^{n-3} \times \mathbb{Z}$ . Let

$$\phi: \mathbb{R}^{n-3} \times \mathbb{R} \rightarrow \mathbb{R}^{n-3} \times \mathbb{R}, (\nu_2, \dots, \nu_{n-2}, d) \mapsto (u(2), \dots, u(n-2), d),$$

where  $u(\ell) = 2(\nu_2 + \dots + \nu_\ell) - d(w_2 + \dots + w_\ell - w_1)$  for  $2 \leq \ell \leq n-2$ . Then  $\phi$  has an inverse given by letting

$$\nu_2 = \frac{u(2) - dw_1 + dw_2}{2} \text{ and } \nu_\ell = \frac{u(\ell) - u(\ell-1) + dw_\ell}{2}$$

for  $3 \leq \ell \leq n-2$ . So  $\phi$  is an isomorphism. Moreover, it induces an isomorphism between  $\mathbb{Z}^{n-3} \times \mathbb{Z}$  and  $M \times \mathbb{Z}$ .

We claim that  $\phi(dQ_w \times \{d\}) = dP_w \times \{d\}$ . Using the fact that for  $(\nu_1, \dots, \nu_n) \in dQ_w$  we have  $\nu_1 = dw_1, \nu_n = 0$ , and  $\nu_{n-1} = \frac{d|w|}{2} - (dw_1 + \nu_2 + \dots + \nu_{n-2})$ , the inequalities for  $dQ_w \cap \mathbb{R}^{n-3}$  are in the left column below, where  $3 \leq \ell \leq n-2$ . The corresponding inequalities for  $dP_w$  are on the right.

$$\begin{array}{ll} \nu_2 \geq 0 & u(2) \geq dw_1 - dw_2 \\ \nu_2 \leq dw_2 & u(2) \leq dw_1 + dw_2 \\ \nu_2 \geq dw_2 - dw_1 & u(2) \geq dw_2 - dw_1 \\ \nu_\ell \geq 0 & u(\ell-1) - u(\ell) \leq dw_\ell \\ \nu_\ell \leq dw_\ell & u(\ell) - u(\ell-1) \leq dw_\ell \\ 2(\nu_2 + \dots + \nu_{\ell-1}) + \nu_\ell \geq d(w_2 + \dots + w_\ell - w_1) & u(\ell) + u(\ell-1) \geq dw_\ell \\ \nu_2 + \dots + \nu_{n-2} \leq \frac{d|w|}{2} - dw_1 & u(n-2) \leq dw_{n-1} + dw_n \\ \nu_2 + \dots + \nu_{n-2} \geq \frac{d|w|}{2} - dw_1 - dw_{n-1} & u(n-2) \geq dw_n - dw_{n-1} \\ \nu_2 + \dots + \nu_{n-2} \geq \frac{d|w|}{2} - dw_1 - dw_n & u(n-2) \geq dw_{n-1} - dw_n \end{array}$$

where the last inequality on the left follows from (2.3) for  $\ell = n-1$ . It is now easy to check that the inequalities on the left correspond to the inequalities on the right under  $\phi$ , so  $\phi$  induces the required isomorphism of semigroups.  $\square$

**Lemma 5.4.** *We have the following properties of  $P_w$ .*

- (i) *The polytope  $P_w$  is normal with respect to  $M$ , i.e., every lattice point in  $mP_w \cap M$  is a sum of  $m$  lattice points in  $P_w \cap M$ .*
- (ii) *For  $v, v' \in P_w \cap M$  there is  $u, u' \in P_w \cap M$  such that  $v + v' = u + u'$  and  $|u(i) - u'(i)| \leq 2$  for all  $2 \leq i \leq n-2$ .*

*Proof.* The proof of (i) closely follows [16, Lemma 7.3] and is essentially the proof of Lemma 6.4 in [15]. We first need to introduce some notation. Let  $\sigma \in \{+, -\}$  and let  $e^\sigma$  denote rounding to the nearest even integer, where for  $a \in \mathbb{Z}$ , we let  $e^+(2a+1) = 2a+2$

and  $e^-(2a+1) = 2a$ . For  $r = (r(2), \dots, r(n-2)) \in mP_w \cap M$ , we say that a sequence of signs  $\sigma(i) \in \{+, -\}$  for  $2 \leq i \leq n-3$  is  $(r, m)$ -admissible if it satisfies  $\sigma(i+1) = -\sigma(i)$  if and only if  $\frac{r(i)}{m}$  and  $\frac{r(i+1)}{m}$  are odd integers and  $\frac{r(i)}{m} + \frac{r(i+1)}{m} = w_{i+1}$ . Such a sequence exists and is unique up to a global sign change.

We claim that if  $\sigma(i)$  is  $(r, m)$ -admissible, then

$$u_r = (u(2), \dots, u(n-2)) = \left( e^{\sigma(2)} \left( \frac{r(2)}{m} \right), \dots, e^{\sigma(n-2)} \left( \frac{r(n-2)}{m} \right) \right),$$

is a lattice point in  $P_w$ .

The following properties of  $e^\sigma$  for  $\sigma \in \{+, -\}$ ,  $a \in 2\mathbb{Z}$  and  $x, y \in \mathbb{R}$  will be useful:

- (1)  $e^\sigma$  is increasing.
- (2)  $e^\sigma(x+a) = e^\sigma(x) + a$ .
- (3) If  $a \geq x$  then  $a \geq e^\sigma(x)$ .
- (4) If  $x+y \geq a$  then  $e^\sigma(x) + e^{-\sigma}(y) \geq a$ .
- (5)  $e^\sigma(x) + e^\sigma(y) \geq x+y-2$ .
- (6)  $e^\sigma(-x) = -e^{-\sigma}(x)$ .

That  $(w_1, w_2, u(2))$  and  $(u(n-2), w_{n-1}, w_n)$  satisfy the triangle inequalities follows from the assumption that  $(w_1, w_2, \frac{r(2)}{m})$  and  $(\frac{r(n-2)}{m}, w_{n-1}, w_n)$  satisfy the triangle inequalities and (1), (2) and (3). For example, we have  $w_1 + u(2) = w_1 + e^\sigma\left(\frac{r(2)}{m}\right) = e^\sigma\left(w_1 + \frac{r(2)}{m}\right) \geq e^\sigma(w_2) = w_2$  and  $w_1 + w_2 \geq e^\sigma\left(\frac{r(2)}{m}\right) = u(2)$ .

When  $2 \leq i \leq n-3$ , we have to show

$$(5.1) \quad u(i) + w_{i+1} \geq u(i+1)$$

$$(5.2) \quad u(i+1) + w_{i+1} \geq u(i)$$

$$(5.3) \quad u(i) + u(i+1) \geq w_{i+1}.$$

We consider two cases. We first assume that  $\sigma(i) = \sigma(i+1)$ , and we let  $\sigma = \sigma(i) = \sigma(i+1)$ . Then  $\frac{r(i)}{m}$  and  $\frac{r(i+1)}{m}$  are not both odd integers or  $\frac{r(i)}{m} + \frac{r(i+1)}{m} > w_{i+1}$ . To see (5.2), note that  $u(i+1) + w_{i+1} = e^\sigma\left(\frac{r(i+1)}{m}\right) + w_{i+1} = e^\sigma\left(\frac{r(i+1)}{m} + w_{i+1}\right) \geq e^\sigma\left(\frac{r(i)}{m}\right) = u(i)$  by (1), (2) and the assumption that  $(\frac{r(i)}{m}, \frac{r(i+1)}{m}, w_{i+1})$  satisfy the triangle inequalities. (5.1) follows similarly. For (5.3), if both  $\frac{r(i)}{m}$  and  $\frac{r(i+1)}{m}$  are odd integers then  $\frac{r(i)}{m} + \frac{r(i+1)}{m} > w_{i+1}$  by assumption. Hence, by (5),  $u(i) + u(i+1) \geq \frac{r(i)}{m} + \frac{r(i+1)}{m} - 2 > w_{i+1} - 2$  which implies (5.3) since  $u(i), u(i+1)$  and  $w_{i+1}$  are even. Otherwise there exists  $x \in \left\{ \frac{r(i)}{m}, \frac{r(i+1)}{m} \right\}$  that is not an odd integer. Then  $e^\sigma(x) = e^{-\sigma}(x)$  and now (5.3) follows from (4).

Suppose now that  $\sigma(i+1) = -\sigma(i)$ . Then  $\frac{r(i)}{m}$  and  $\frac{r(i+1)}{m}$  are odd integers and  $\frac{r(i)}{m} + \frac{r(i+1)}{m} = w_{i+1}$ . Then (5.3) follows from (4). If  $\sigma(i) = +$  and  $\sigma(i+1) = -$  then (5.1) follows easily. Since  $\frac{r(i+1)}{m}$  is a positive odd integer, we have  $2\frac{r(i+1)}{m} > 0$ , and using the assumption  $\frac{r(i)}{m} + \frac{r(i+1)}{m} = w_{i+1}$ , we obtain  $u(i+1) + w_{i+1} = \frac{r(i+1)}{m} - 1 + w_{i+1} > \frac{r(i)}{m} - 1 =$

$u(i) - 2$ . Since  $u(i+1), u(i)$  and  $w_{i+1}$  are even integers, (5.2) follows. The case  $\sigma(i) = -$  and  $\sigma(i+1) = +$  follows analogously.

To show (i), we proceed by induction on  $m$ . For  $m = 1$  there is nothing to show. Assume that  $m \geq 2$ . For  $r \in mP \cap M$ , let  $u = u_r$  as in the claim. Then by the claim,  $u$  lies in  $P_w \cap M$ . Let  $v = (v(2), \dots, v(n-2))$ , where  $v(i) = e^{-\sigma(i)} \left( \frac{m-1}{m} r(i) \right)$ . Note that it follows from (4) and (6) that  $u(i) + v(i) = r(i)$ , so  $u + v = r$ . Replacing  $u$  by  $v$ ,  $\sigma$  by  $-\sigma$ ,  $w_i$  by  $(m-1)w_i$  and  $\frac{r(i)}{m}$  by  $\frac{m-1}{m} r(i)$ , and noting that  $\left( \frac{m-1}{m} r \right) \in (m-1)P_w$  and that  $\frac{r(i)}{m}$  is odd if and only if  $\frac{(m-1)r(i)}{m}$  is odd, the same arguments as in the proof of the claim show that  $v = r - u \in (m-1)P_w \cap M$ . But  $v$  is a sum of  $m-1$  lattice points in  $P_w$  by induction.

For (ii), we apply the claim to  $r = v + v'$  and let  $(\sigma(2), \dots, \sigma(n-2))$  be a  $(r, 2)$ -admissible sequence of signs. Then by the claim we have that

$$\begin{aligned} u &= \left( e^{\sigma(2)} \left( \frac{r(2)}{2} \right), \dots, e^{\sigma(n-2)} \left( \frac{r(n-2)}{2} \right) \right) \text{ and} \\ u' &= \left( e^{-\sigma(2)} \left( \frac{r(2)}{2} \right), \dots, e^{-\sigma(n-2)} \left( \frac{r(n-2)}{2} \right) \right) \end{aligned}$$

are lattice points in  $P_w$ . The assertion follows.  $\square$

Let  $J$  be the toric ideal associated to the polytope  $P_w$ , i.e.,  $J$  is the kernel of the map  $\mathbf{k}[X_u \mid u \in P_w \cap M] \rightarrow \mathbf{k}[S_{P_w}]$ , where  $X_u \mapsto (u(2), u(3), \dots, u(n-2), 1)$ . Since  $P_w$  is normal, the line bundle associated to  $P_w$  induces a projectively normal embedding of the toric variety  $X_{P_w}$  associated to  $P_w$ , with homogeneous coordinate ring  $\mathbf{k}[X_u]/J$ . By Proposition 4.5 and Lemma 5.3, the toric variety  $X_{P_w}$  is isomorphic to  $\text{Proj}(\text{in}_{\prec}(R))$ .

**Definition 5.5.** Let  $m = \prod_{t=1}^{\ell} X_{u_t}$  be a monomial in  $\mathbf{k}[X_u]$ . We define the *norm* of  $m$  to be

$$N(m) = \sum_{t=1}^{\ell} \|u_t\|^2 = \sum_{t=1}^{\ell} \sum_{i=2}^{n-2} u_t(i)^2.$$

**Definition 5.6.** We say that a monomial  $m$  is *norm-minimal*, if for all  $m'$  with  $m' - m \in J$ , we have  $N(m') \geq N(m)$ .

The following lemma characterizes norm-minimal monomials.

**Lemma 5.7.** *A monomial  $m$  is norm-minimal if and only if for all  $X_v X_{v'}$  dividing  $m$ , we have  $|v(i) - v'(i)| \leq 2$  for all  $2 \leq i \leq n-2$ .*

*Proof.* To prove the Lemma, we will need the following fact:

- ( $\star$ ) Let  $a_i, b_i \in 2\mathbb{Z}$  with  $a_1 + \dots + a_{\ell} = b_1 + \dots + b_{\ell}$ , and  $|a_i - a_j| \leq 2$  for all  $i, j$ . Then  $\sum_{i=1}^{\ell} a_i^2 \leq \sum_{i=1}^{\ell} b_i^2$ . Moreover, equality holds if and only if  $\{a_1, \dots, a_{\ell}\} = \{b_1, \dots, b_{\ell}\}$ .

Given ( $\star$ ), assume that  $m$  is norm-minimal, but that there exists a quadratic factor  $X_v X_{v'}$  of  $m$  and  $2 \leq i \leq n-2$  such that  $|v(i) - v'(i)| > 2$ . By Lemma 5.4, there are

$u, u' \in P_w \cap M$  with  $v + v' = u + u'$  and  $|u(i) - u'(i)| \leq 2$  for all  $i$ . So  $X_u X_{u'} - X_v X_{v'} \in J$ , and by  $(\star)$ ,  $N(X_u X_{u'}) < N(X_v X_{v'})$ . Thus for  $m' = m \frac{X_u X_{u'}}{X_v X_{v'}}$ , we have  $m' - m \in J$ , but  $N(m') < N(m)$ , a contradiction. The converse follows immediately from  $(\star)$ .

For  $(\star)$ , note that since  $|a_i - a_j| \leq 2$  for all  $i, j$ , there exists  $\alpha$  such that  $a_i \in \{\alpha, \alpha + 2\}$  for all  $i$ . After renumbering, we may assume  $a_1 = \dots = a_p = \alpha$  and  $a_{p+1} = \dots = a_\ell = \alpha + 2$ . If we set  $b_i = a_i + k_i$ , then  $\sum k_i = 0$ , and we have  $\sum_{i=1}^\ell b_i^2 - \sum_{i=1}^\ell a_i^2 = \sum_{i=1}^\ell k_i^2 + \sum_{i=p+1}^\ell 4k_i$ . Note that when  $k_i \leq -4$ , then  $k_i^2 + 4k_i \geq 0$ , and if  $k_i \geq 0$ , then  $4k_i \geq 0$ , so it suffices to show that if  $\sum k_i \geq 0$ ,  $k_i \in 2\mathbb{Z}$ , and  $k_i = -2$  for  $p+1 \leq i \leq \ell$ , then  $\sum_{i=1}^\ell k_i^2 + \sum_{i=p+1}^\ell 4k_i \geq 0$ . This in turn follows from the fact that for  $k \in 2\mathbb{N}$  we have  $k^2 \geq 2k$ , so for  $k_i \in 2\mathbb{N}$  with  $\sum_{i=1}^p k_i \geq 2(\ell - p)$ , we have  $\sum_{i=1}^p k_i^2 \geq \sum_{i=1}^p 2k_i \geq 4(\ell - p)$ . Now suppose equality holds, so  $\sum_{i=1}^\ell k_i^2 + \sum_{i=p+1}^\ell 4k_i = 0$  where  $k_i \in 2\mathbb{Z}$  and  $\sum k_i = 0$ . Note that  $k_i^2 + 4k_i$  is non-negative unless  $k_i = -2$ . If  $R = \{i \mid k_i = -2\}$  and  $T = \{i \mid k_i > 0\}$ , then we must have  $\sum_{i \in T} k_i \geq 2|R|$  and  $\sum_{i \in T} k_i^2 + \sum_{i \in T, i \geq p+1} 4k_i \leq 4|R|$ , but the only situation when this holds is when  $k_i = 2$  for all  $i \in T$ , no element in  $T$  is larger than  $p$ , and  $|T| = |R|$ . The claim follows.  $\square$

We now proceed to define a term order on the monomials in the variables  $X_u$ ,  $u \in P_w$ .

We first use the standard lexicographic ordering  $<_{\text{lex}}$  on  $M \cong \mathbb{Z}^{n-3}$  to order the variables  $X_u, u \in P_w$ . Let  $\prec_{\text{grevlex}}$  be the graded reverse lexicographic order on  $k[X_u \mid u \in P_w \cap M]$  induced by this ordering of the variables, i.e.,  $m' \prec_{\text{grevlex}} m$  if  $\deg(m') < \deg(m)$  or  $\deg(m') = \deg(m)$  and for the smallest variable where the exponents of  $m$  and  $m'$  differ the exponent of  $m'$  is larger than the exponent of  $m$ .

We define  $m' \prec m$  iff

- $\deg(m') < \deg(m)$ , or
- $\deg(m') = \deg(m)$  and  $N(m') < N(m)$ , or
- $\deg(m') = \deg(m)$ ,  $N(m') = N(m)$ , and  $m' \prec_{\text{grevlex}} m$ .

We shall consider two types A, B of quadratic binomial relations.

**Definition 5.8.** (Type A) The type A relations are relations  $X_v X_{v'} - X_u X_{u'} \in J$ , where  $N(X_v X_{v'}) > N(X_u X_{u'})$ .

**Definition 5.9.** (Type B at position  $j$ ) Suppose that  $u, v \in P_w \cap M$ , and  $3 \leq j \leq n - 2$ . Suppose that  $(u(j-1), w_j, v(j))$  and  $(v(j-1), w_j, u(j))$  satisfy the triangle inequalities. Let

$$\begin{aligned} u' &= (u(2), \dots, u(j-1), v(j), \dots, v(n-2)), \\ v' &= (v(2), \dots, v(j-1), u(j), \dots, u(n-2)). \end{aligned}$$

We call  $X_u X_v - X_{u'} X_{v'}$  a type B relation at position  $j$ .

Note that  $u', v' \in P_w$  and  $u + v = u' + v'$ , so a type B relation is well-defined. Moreover,  $N(X_u X_v) = N(X_{u'} X_{v'})$  for any relation  $X_u X_v - X_{u'} X_{v'}$  of type B.



**Theorem 5.10.** *The relations of type A and B form a quadratic Gröbner basis for the ideal  $J$ .*

*Proof.* It suffices to show that

$$\text{in}_{\prec}(J) = \langle \text{in}_{\prec} f \mid f \text{ is a type A or a type B relation} \rangle.$$

Let  $m \in \text{in}_{\prec} J$ . Suppose  $m$  is not norm-minimal. By Lemma 5.7 there exists a quadratic factor  $X_v X_{v'}$  dividing  $m$  such that  $|v(i) - v'(i)| > 2$  for some  $i$ . By Lemma 5.4 there is  $u, u' \in P_w \cap M$  such that  $X_v X_{v'} - X_u X_{u'} \in J$ , and  $|u(i) - u'(i)| \leq 2$  for all  $i$ . By Lemma 5.7,  $X_u X_{u'}$  is norm-minimal, and  $X_v X_{v'}$  is not, so  $X_v X_{v'} - X_u X_{u'}$  is a type A relation and  $m \in \langle X_v X_{v'} \rangle \subset \langle \text{in}_{\prec} f \mid f \text{ is a type A relation} \rangle$ .

Suppose  $m$  is norm-minimal. Let  $f \in J$  be such that  $\text{in}_{\prec} f = m$ . Since  $J$  is a homogeneous binomial ideal, we may assume that  $f = m - m'$ , where  $m'$  is a norm-minimal monomial of the same degree as  $m$ . Then  $N(m') = N(m)$ , but  $m' \prec_{\text{grevlex}} m$ . Let  $m = X_{u_1} X_{u_2} \cdots X_{u_\ell}$ , where  $u_1 \leq_{\text{lex}} u_2 \leq_{\text{lex}} \cdots \leq_{\text{lex}} u_\ell$  and  $m' = X_{v_1} X_{v_2} \cdots X_{v_\ell}$  where  $v_1 \leq_{\text{lex}} v_2 \leq_{\text{lex}} \cdots \leq_{\text{lex}} v_\ell$ . Note that since  $m - m' \in J$ , we have

$$(5.4) \quad \sum_{s=1}^{\ell} u_s = \sum_{s=1}^{\ell} v_s.$$

Factoring out the largest common multiple of  $m$  and  $m'$ , we may assume  $v_1 <_{\text{lex}} u_1$ . Let  $j$  be the first index where  $v_1(j) < u_1(j)$ . Note that if  $i < j$  then  $u_1(i) = v_1(i)$ .

It follows from Lemma 5.7 and from (5.4) that for every  $2 \leq k \leq n - 2$  there exists an even integer  $\alpha_k$  such that for every  $X_u$  dividing  $m$  or  $m'$ ,  $u(k) \in \{\alpha_k, \alpha_k + 2\}$ .

This implies that  $v_1(j) = \alpha_j$  and  $u_1(j) = \alpha_j + 2$ . It follows from (5.4) that there exists some  $t > 1$  such that  $u_t(j) = v_1(j) = \alpha_j$ . Note that, since  $u_1 <_{\text{lex}} u_t$  and  $u_1(j) > u_t(j)$ , there is  $i' < j$  such that  $u_1(i') < u_t(i')$  and  $u_1(i) = u_t(i)$  for  $i < i'$ . In particular,  $j > 2$ . Now  $(u_1(j-1), w_j, u_t(j)) = (v_1(j-1), w_j, v_1(j))$  satisfy the triangle inequalities, since  $v_1 \in P_w$ . If  $(u_t(j-1), w_j, u_1(j))$  satisfy the triangle inequalities, then there exists a type B relation at position  $j$  of the form  $X_{u_1} X_{u_t} - X_{u'_1} X_{u'_t}$ . Since  $u'_1 <_{\text{lex}} u_1 <_{\text{lex}} u_t$ , we have  $X_{u_1} X_{u_t} \succ_{\text{grevlex}} X_{u'_1} X_{u'_t}$ . Therefore,  $m \in \text{in}_{\prec} \{f \mid f \text{ is a type B relation}\}$ .

Now suppose  $(u_t(j-1), w_j, u_1(j))$  do not satisfy the triangle inequalities. This implies that whenever  $v_s(j-1) = u_t(j-1)$ , then  $v_s(j) \neq u_1(j)$ , since  $(v_s(j-1), w_j, v_s(j))$  satisfy the triangle inequalities and so  $v_s(j) = v_1(j)$ . Note however that  $v_1(j-1) = u_1(j-1) \neq u_t(j-1)$  by assumption. In particular,

$$\{s \mid 1 \leq s \leq \ell, v_s(j-1) = u_t(j-1)\} \subsetneq \{s \mid 1 \leq s \leq \ell, v_s(j) = v_1(j)\}.$$

Since  $\sum u_s(j-1) = \sum v_s(j-1)$ , we have

$$\#\{s \mid 1 \leq s \leq \ell, v_s(j-1) = u_t(j-1)\} = \#\{s \mid 1 \leq s \leq \ell, u_s(j-1) = u_t(j-1)\}$$

and similarly

$$\#\{s \mid 1 \leq s \leq \ell, v_s(j) = v_1(j)\} = \#\{s \mid 1 \leq s \leq \ell, u_s(j) = v_1(j)\}.$$

Using  $u_t(j) = v_1(j)$ , we obtain

$$\#\{s \mid 1 \leq s \leq \ell, u_s(j-1) = u_t(j-1)\} < \#\{s \mid 1 \leq s \leq \ell, u_s(j) = u_t(j)\}.$$

Pick an  $s$  such that  $u_s(j-1) \neq u_t(j-1)$  but  $u_s(j) = u_t(j)$ . Since  $u_1(j) \neq u_t(j)$ ,  $s \neq 1$ . By assumption,  $u_1(j-1) \neq u_t(j-1)$ , hence  $u_s(j-1) = u_1(j-1)$  since  $\{u_s(j-1), u_1(j-1), u_t(j-1)\} \subset \{\alpha_{j-1}, \alpha_{j-1} + 2\}$ . So  $(u_s(j-1), w_j, u_1(j)) = (u_1(j-1), w_j, u_1(j))$  satisfy the triangle inequalities. On the other hand, since  $u_s(j) = u_t(j) = v_1(j)$ , we have that  $(u_1(j-1), w_j, u_s(j)) = (v_1(j-1), w_j, v_1(j))$  satisfy the triangle inequalities. So there exists a type B relation  $X_{u_1}X_{u_s} - X_{u'_1}X_{u'_s}$  at position  $j$ . Note that  $X_{u_1}X_{u_s} \succ_{\text{grevlex}} X_{u'_1}X_{u'_s}$ , so  $m \in \text{in}_{\prec}\{f \mid f \text{ is a type B relation}\}$ .  $\square$

**Proposition 5.11.** *The initial ideal  $\text{in}_{\prec}(J)$  is radical.*

*Proof.* Since the initial ideal is generated by quadratic monomials, we only need to check there is no perfect square  $X_v^2 \in \text{in}_{\prec}(J)$ . Suppose that  $X_v^2 - X_uX_w \in J$  is a non-zero relation for some  $u, v, w$ . Then  $2v = u + w$ , and so  $N(X_v^2) < N(X_uX_w)$ , since by Lemma 5.7  $X_v^2$  is norm-minimal, but  $X_uX_w$  is not. So  $X_v^2$  is no leading term of any binomial in  $J$ .  $\square$

Note that this implies that  $P_w$  admits a regular unimodular triangulation, see [35, Corollary 8.9]

*Proof of Theorem 1.2.* By Kempe's theorem [16, Theorem 2.3],  $R_w$  is generated by its lowest degree invariants. Note that for  $w \in (2\mathbb{Z})^n$  this also follows from Lemma 5.4, Proposition 4.5, and Lemma 5.3. Recall that  $I_w$  denotes the ideal of relations between these generators. Since the polytopal semigroup algebra  $\mathbf{k}[S_{P_w}]$  is a SAGBI degeneration of  $R_w$  by Proposition 4.5, and since the toric ideal  $J$  associated to  $P_w$  has a square free quadratic initial ideal by Theorem 5.10 and Proposition 5.11, the claim follows from [4, Corollary 2.2]. The Koszul property follows for example from [8, Proposition 3].  $\square$

**Remark 5.12.** In [24, Theorem 1.10] Manon generalizes Theorem 1.2 to certain subpolytopes of  $P_w$ . For  $L \geq 0$ , the polytope  $P_L$  is given by the adding to the inequalities of Definition 5.1 the inequalities  $w_1 + w_2 + u(2) \leq 2L, u(n-2) + w_{n-1} + w_n \leq 2L, u(i-1) + w_i + u_i \leq 2L$ . Note that when  $L$  is large, then  $P_w = P_L$ . For  $L = 1$ , the polytopes  $P_L$  are slices of the polytopes studied in [3].

**Remark 5.13.** It is easy to see that for  $w = 1^n$ ,  $P_w$  is reflexive and that  $S_{P_w}$  is Gorenstein. This implies that the toric variety  $V$  associated to  $P_w$  is arithmetically Gorenstein and Fano. In particular,  $V$  has canonical singularities. However, when  $w = 1^6$ , then  $V$  does not have terminal singularities. Compare also [30, Proposition 1.4]. The Gorenstein property for the toric varieties arising as degenerations of  $R_w$  corresponding to arbitrary trivalent trees was studied by Manon in [25].

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